Vector Fields, Work, Circulation, and Flux

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Overview

When we study physical phenomena that are represented by vectors, we replace integrals over closed intervals by integrals over paths through vector fields. Gravitational and electric forces have both a direction and a magnitude. They are represented by a vector at each point in their domain, producing a vector field.

We use integrals to find the work done in moving an object along a path against a variable force (such as a vehicle sent into space against Earch's gravitational field) or to find the work done by a vector field in moving an object along a path through the field (such as the work done by an accelerator in raising the energy of a particle).

We also use line integrals to find the rates at which fluids flow along and across curves.

Vector Fields

Suppose a region in the plane or in space is occupied by a moving fluid such as air or water. Imagine that the fluid is made up of a very large number of particles, and that at any instant of time a particle has a velocity $\boldsymbol{v}$.

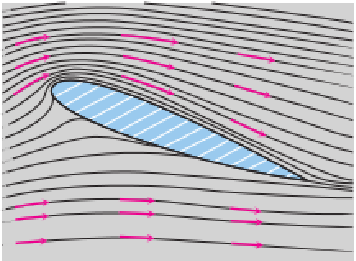
If we take a picture of the velocities of some particles at different position points at the same instant, we would expect to find that these velocities vary from position to position.

We can think of a velocity vector as being attached each point of the fluid. Such a fluid flow exemplifies a **vector field**.

Vector Fields

For example, the following figure shows a velocity vector field obtained by attaching a velocity vector to each point of air flowing around an airfoil in a wind tunnel.

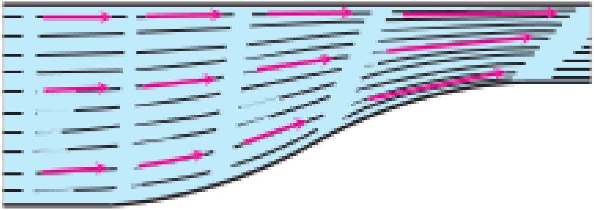
The streamlines are made visible by kerosene smoke.



Vector Fields

The figure shows another “vector field of velocity vectors” along the streamlines of water moving through a contracting channel.

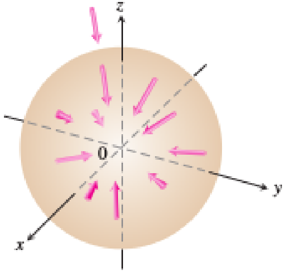
The water speeds up as the channel narrows and the velocity vectors increases in length.



Vector Fields

In addition to vector fields associated with fluid flows, there are vector force fields that are associated with gravitational attaction, magnetic force fields, electric fields, and even purely mathematical fields.

Vectors in a gravitational field point toward the center of mass that gives the source of the field.



Vector Fields

Generally, a **vector field** on a domain in the plane or in space is a function that assigns a vector to each point in the domain.

A field of three-dimensional vectors might have formula like

$$\boldsymbol{F}(x, y, z) = M(x, y, z)\boldsymbol{i} + N(x, y, z)\boldsymbol{j} + P(x, y, z)\boldsymbol{k}.$$

The field is **continuous** if the **component functions** M, N, and P are continuously **differentiable** if M, N, and P are differentiable, and so on.

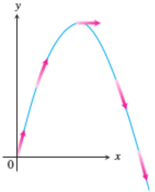
A field of two-dimensional vectors might have a formula like

$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}.$$

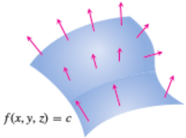
If we attach a projectile's velocity vector to each point of the projectile's trajectory in the plane of motion, we have a two-dimensional field defined aloing the trajectory.

If we attach the gradient vector of a scalar function to each point of a level surface of the functions, we have a three-dimensional field on the surface. If we attach the velocity vector to each point of a flowing fluid, we have a three-dimensional field defined on a region in space. These and other fields are illustrated in the following figures.

The velocity vectors $\boldsymbol{v}(t)$ of a projectile's motion make a vector field along the trajectory.



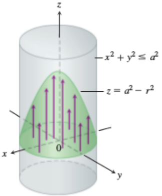
The field of gradient vectors ∇f on a surface $f(x, y, z) = c$, shown below.



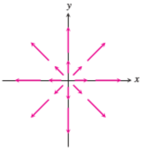
The flow of fluid in a long cylindrical pipe : The vectors

$$\boldsymbol{v} = (a^2 - r^2)\boldsymbol{k}$$

inside the cylinder that have their bases in the xy-plane have their tips on the paraboloid $z = a^2 - r^2$.



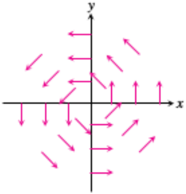
The radial field $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$ of position vectors of points in the plane, shown below.



The circumferential or "spin" field of unit vectors

$$\boldsymbol{F} = (-y\boldsymbol{i} + x\boldsymbol{j})/(x^2 + y^2)^{1/2}$$

in the plane. The field is not defined at the origin.



Vector Fields

To sketch the fields that had formulas, we picked a representative selection of the main points and sketched the vectors attached to them.

Convention : The arrows representing the vectors are drawn with their tails, not their heads, at the points where the vector functions are evaluated.

This is different from the way we draw position vector of planets and projectiles, with their tails at the origin and their heads at the planet's and projectile's locations.

Gradient Field

The **gradient field** of a differentiable function $f(x, y, z)$ is the field of gradient vectors

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}.$$

Gradient fields are of special importance in engineering, mathematics, and physics.

Exercise 1.

1. *Find the gradient field of* $f(x, y, z) = xyz$.
2. *Find the gradient field of* $g(x, y, z) = e^z - \ln(x^2 + y^2)$.

Work Done by a Force over a Curve in Space

Suppose that the vector field

$$\boldsymbol{F} = M(x, y, z)\boldsymbol{i} + N(x, y, z)\boldsymbol{j} + P(x, y, z)\boldsymbol{k}.$$

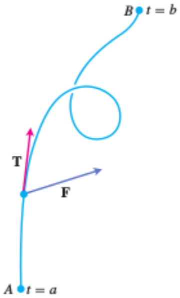
represents a force throughout a region in space (it might be the force or gravity or an electromagnetic force of some kind) and that

$$\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}, \quad a \leq t \leq b,$$

is a smooth curve in the region.

Then the integral of $\boldsymbol{F}.\boldsymbol{T}$, the scalar component of $\boldsymbol{F}$ in the direction of the curve's unit tangent vector, over the curve is called the work done by $\boldsymbol{F}$ over the curve from a and b.

Work Done by a Force over a Curve in Space



The work done by a force $\boldsymbol{F}$ is the line integral of the scalar component $\boldsymbol{F}.\boldsymbol{T}$ over the smooth curve from A to B.

Work Done by a Force over a Curve in Space

Definition 2.

The work *done by a force*

$$\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}$$

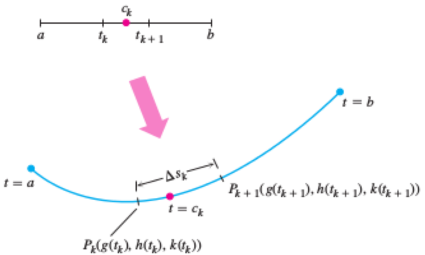
over a smooth curve $\boldsymbol{r}(t)$ *from* $t = a$ *to* $t = b$ *is*

$$W = \int_{t=a}^{t=b} \boldsymbol{F}.\boldsymbol{T}\ ds.$$

We divide the curve into short segments, apply the (constant-force)$\times$ (distance) formula for work to approximate the work over each curved segment, add the results to approximate the work over the entire curve, and calculate the work as the limit of the approximating sums as the segments become shorter and more numerous.

To find exactly what the limiting integral should be, we partition the parameter interval $[a, b]$ in the usual way the choose a point c_k in each subinterval $[t_k, t_{k-1}]$.

The partition of $[a, b]$ determines ("induces," we say) a partition of the curve, with the point P_k being the tip of the position vector $\boldsymbol{r}(t_k)$ and Δs_k being the length of the curve segment $P_k P_{k-1}$.

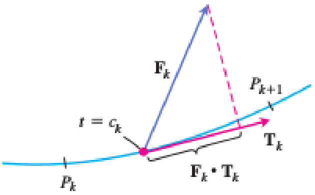


Each partition of $[a, b]$ induces a partition of the curve

$$\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}.$$

If $\boldsymbol{F}_k$ denotes the value of $\boldsymbol{F}$ at the point on the curve corresponding to $t = c_k$ and $\boldsymbol{T}_k$ denotes the curve's unit tangent vector at this point, then $\boldsymbol{F}_k.\boldsymbol{T}_k$ is the scalar component of $\boldsymbol{F}$ in the direction of $\boldsymbol{T}$ at $t = c_k$.

Work Done by a Force over a Curve in Space



An enlarged view of the curve segment $P_k P_{k-1}$, showing the force and the unit tangent vectors at the point on the curve where $t = c_k$.

The work done by $\boldsymbol{F}$ along the curve segment $P_k P_{k-1}$ is approximately

$$\Big(\text{Force component in direction of motion} \quad \times \quad \text{distance applied}\Big) \qquad = \boldsymbol{F}_k.\boldsymbol{T}_k \Delta s_k.$$

Work Done by a Force over a Curve in Space

The work done by $\boldsymbol{F}$ along the curve from $t = a$ to $t = b$ is approximately

$$\sum_{k=1}^{n} \boldsymbol{F}_k . \boldsymbol{T}_k \Delta s_k .$$

As the norm of the partition of $[a, b]$ approaches zero, the norm of the induced partition of the curve approaches zero and theses sums approach the line integral

$$\int_{t=a}^{t=b} \boldsymbol{F} . \boldsymbol{T} \; ds.$$

The sign of the number we calculate with this integral depends on the direction in which the curve is traversed as t increases. If we reverse the direction of motion, we reverse the direction of $\boldsymbol{T}$ and change the sign of $\boldsymbol{F} . \boldsymbol{T}$ and its integral.

Work Done by a Force over a Curve in Space

The table given below shows six ways to write the work integral.

In the table,

$$\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

is a smooth curve, and

$$d\boldsymbol{r} = \frac{d\boldsymbol{r}}{dt}dt = dg\boldsymbol{i} + dh\boldsymbol{j} + dk\boldsymbol{k}$$

is its differential.

Six Different Ways to Write the Work Integrals

$$
\begin{aligned}
\boldsymbol{W} &= \int_{t=a}^{t=b} \boldsymbol{F}.\boldsymbol{T}\ ds \\
&= \int_{t=a}^{t=b} \boldsymbol{F}.d\boldsymbol{r} \\
&= \int_a^b \boldsymbol{F}.\frac{d\boldsymbol{r}}{dt}dt \\
&= \int_a^b \left(M\frac{dg}{dt} + N\frac{dh}{dt} + P\frac{dk}{dt}\right)dt \\
&= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right)dt \\
&= \int_a^b Mdx + Ndy + Pdz.
\end{aligned}
$$

Evaluating a Work Integral

To evaluate the work integral along a smooth curve $\boldsymbol{r}(t)$, take these steps:

1. Evaluate $\boldsymbol{F}$ on the curve as a function of the parameter t.
2. Find $d\boldsymbol{r}/dt$.
3. Integrate $\boldsymbol{F}.d\boldsymbol{r}/dt$ from $t = a$ to $t = b$.

Flow Integrals and Circulation for Velocity Fields

Instead of being a force field, suppose that $\boldsymbol{F}$ represents the velocity field of a fluid flowing through a region in space (a tidal basin or the turbine chamber of a hydroelectric generator, for example).

Under these circumstances, the integral of $\boldsymbol{F}.\boldsymbol{T}$ along a curve in the region gives the fluid's flow along the cuve.

Flow Integral, Circulation

Definition 3.

If $\boldsymbol{r}(t)$ *is a smooth curve in the domain of a continuous velocity field* $\boldsymbol{F}$*, the* flow *along the curve from* $t = a$ *to* $t = b$ *is*

$$Flow = \int_a^b \boldsymbol{F}.\boldsymbol{T}\ ds.$$

The integral in this case is called a flow integral*. If the curve is a closed loop, the flow is called the* circulation *around the curve.*

We evaluate flow integrals the same way we evaluate work integrals.

Flux Across a Plane Curve

To find the rate at which a fluid is entering or leaving a region enclosed by a smooth curve C in the xy-plane, we calculate the line integral over C of $\boldsymbol{F}.\boldsymbol{n}$, the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector.

The value of this integral is the flux of $\boldsymbol{F}$ across C.

If $\boldsymbol{F}$ were an electric field or a magnetic field, for instance, the integral $\boldsymbol{F}.\boldsymbol{n}$ would still be called the flux of the field across C.

Flux Across a Closed Curve in the Plane

Definition 4.

If C is a smooth closed curve in the domain of a continuous vector field

$$\boldsymbol{F} = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$

in the plane, and if $\boldsymbol{n}$ is the outward-pointing unit normal vector on C, the flux *of $\boldsymbol{F}$ across C is*

$$\textit{Flux of } \boldsymbol{F} \textit{across } C = \int_C \boldsymbol{F}.\boldsymbol{n}\ ds.$$

Notice the difference between flux and circulation.

The flux of $\boldsymbol{F}$ across C is the integral with respect to arc length of $\boldsymbol{F}.\boldsymbol{n}$, the scalar component of $\boldsymbol{F}$ in the direction of the outward normal.

Flux Across a Closed Curve in the Plane

The circulation of $\boldsymbol{F}$ around C is the line integral with respect to arc length $\boldsymbol{F}.\boldsymbol{T}$, the scalar component of $\boldsymbol{F}$ in the direction of the unit tangent vector.

Flux is the integral of the normal component of $\boldsymbol{F}$; circulation is the integral of the tangential component of $\boldsymbol{F}$. To evaluate the integral $\int_C \boldsymbol{F}.\boldsymbol{n}\, ds$, we begin with a smooth parameterization

$$x = g(t), \quad y = h(t), \quad a \leq t \leq b,$$

that traces the curve C exactly once as t increases from a to b.

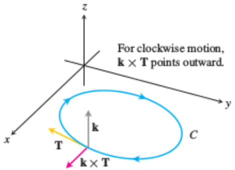
We can find the outward unit normal vector $\boldsymbol{n}$ by crossing the curve's unit tangent vector $\boldsymbol{T}$ with the vector $\boldsymbol{k}$.

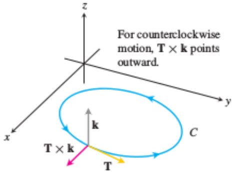
But which order do we choose, $\boldsymbol{T} \times \boldsymbol{k}$ or $\boldsymbol{k} \times \boldsymbol{T}$?

Which one points outward? It depends on which way C is traversed as t increases.

If the motion is clockwise, $\boldsymbol{k} \times \boldsymbol{T}$ points outward; if the motion is counterclockwise, $\boldsymbol{T} \times \boldsymbol{k}$ points outward.

The usual choice is $\boldsymbol{n} = \boldsymbol{T} \times \boldsymbol{k}$, the choice that assumes counterclockwise motion.





Flux Across a Closed Curve in the Plane

Thus, although the value of the arc length integral in the definition of flux

$$\int_C \boldsymbol{F}.\boldsymbol{n}\; ds$$

does not depend on which way C is traversed, the formulas we are about to derive for evaluating the integral in

$$\int_C \boldsymbol{F}.\boldsymbol{n}\; ds$$

will assume counterclockwise motion. In terms of components,

$$\boldsymbol{n} = \boldsymbol{T} \times \boldsymbol{k} = \left(\frac{dx}{ds}\boldsymbol{i} + \frac{dy}{ds}\boldsymbol{j}\right) \times \boldsymbol{k} = \frac{dy}{ds}\boldsymbol{i} - \frac{dx}{ds}\boldsymbol{j}.$$

Flux Across a Closed Curve in the Plane

If $\boldsymbol{F} = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$, then

$$\boldsymbol{F}.\boldsymbol{n} = M(x, y)\frac{dy}{ds} - N(x, y)\frac{dx}{ds}.$$

Hence,

$$\int_C \boldsymbol{F}.\boldsymbol{n}\ ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)ds = \oint_c Mdy - Ndx.$$

We put a directed circle $\circ$ on the last integral as a remainder that the integration around the closed curve C is to be in the counterclockwise direction. To evaluate this integral, we express M, dy, N, and dx in terms of t and integrate from $t = a$ to $t = b$. We do not need to know either $\boldsymbol{n}$ or ds to find the flux.

Calculating Flux Across a Smooth Closed Plane Curve

Flux of $\boldsymbol{F} = M\boldsymbol{i} + n\boldsymbol{j}$ across $C = \oint_C Mdy - Ndx$.

The integral can be evaluated from any smooth parameterization

$$x = g(t), y = h(t), a \leq t \leq b,$$

that traces C counterclockwise exactly once.

Exercise 5.

1. *Define the following terms:*
   (a) *Work done by a force over a smooth curve.*
   (b) *Flow of a velocity field along a smooth curve.*
   (c) *Circulation.*
   (d) *Flux of a vector field across a smooth closed curve in the plane.*
2. *Find the work done by force* F *from* $(0,0,0)$ *to* $(1,1,1)$ *over each of the following paths:*
   (a) $\mathsf{F} = xy\mathsf{i} + yz\mathsf{j} + xz\mathsf{k}$ *over the curved path :* $\mathsf{r}(t) = t\mathsf{i} + t^2\mathsf{j} + t^4\mathsf{k}$, $0 \leq t \leq 1$.
   (b) $\mathsf{F} = (y+z)\mathsf{i} + (z+x)\mathsf{j} + (x+y)\mathsf{k}$ *over the straight line path.*
3. *Find the work done by the force*

$$\mathsf{F} = xy\mathsf{i} + (y-x)\mathsf{j}$$

over the straight line from $(1,1)$ *to* $(2,3)$.

Solution for Exercise 5

2. (a) Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_c \mathbf{F}.\frac{dr}{dt}$.
$\mathbf{F} = t^3\mathbf{i} - t^6\mathbf{j} + t^5\mathbf{k}$ and $\frac{dr}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} =$
$t^3 + 2t^7 + 4t^8 \Rightarrow \int_0^1 (t^3 + 2t^7 + 4t^8)dt = \left[\frac{t^4}{4} + \frac{t^8}{4} + \frac{4}{9}t^9\right]_0^1 = \frac{17}{18}$

   (b) Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_c \mathbf{F} \cdot \frac{dr}{dt}$.
$\mathbf{F} = 2t\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k}$ and
$\frac{dr}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = 6t \Rightarrow \int_0^1 6t \; dt = [3t^2]_0^1 = 3$

3. $\mathbf{r} = (\mathbf{i} + \mathbf{j}) + t(\mathbf{i} + 2\mathbf{j}) = (1 + t)\mathbf{i} + (1 + 2t)\mathbf{j}, 0 \leq t \leq 1,$ and $\mathbf{F} = xy\mathbf{i} + (y - x)\mathbf{j}$
$\Rightarrow \mathbf{F} = (1 + 3t + 2t^2)\mathbf{i} + t\mathbf{j}$ and $\frac{dr}{dt} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = 1 + 5t + 2t^2$
$\Rightarrow$ work $= \int_c \mathbf{F} \cdot \frac{dr}{dt} dt = \int_0^1 (1 + 5t + 2t^2)dt = \left[t + \frac{5}{2}t^2 + \frac{2}{3}t^3\right]_0^1 = \frac{25}{6}$

Exercise 6.

Evaluate the flow integral of the velocity field

$$F = (x + y)\mathrm{i} - (x^2 + y^2)\mathrm{j}$$

along each of the following paths from $(1, 0)$ *to* $(-1, 0)$ *in the* xy*-plane.*

(a) *The upper half of the circle* $x^2 + y^2 = 1$.

(b) *The line segment from* $(1, 0)$ *to* $(0, -1)$ *followed by the line segment from* $(0, -1)$ *to* $(-1, 0)$.

Solution for Exercise 6

(a) $\mathbf{r} = (\cos\ t)\mathbf{i} + (\sin\ t)\mathbf{j}, 0 \le t \le \pi, \text{and} \mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{dr}{dt} = (-\sin\ t)\mathbf{i} + (\cos\ t)\mathbf{j}$ and $\mathbf{F} = (\cos\ t + \sin\ t)\mathbf{i} - (\cos^2 t + \sin^2 t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = -\sin\ t\ \cos\ t - \sin^2 t - \cos\ t \ \Rightarrow \int_c \mathbf{F} \cdot \mathbf{T} ds = \int_0^{\pi} (-\sin\ t\ \cos\ t - \sin^2 t - \cos\ t) dt = \left[-\frac{1}{2}\sin^2 t - \frac{1}{2} + \frac{\sin\ 2t}{4} - \sin\ t \right]_0^{\pi} = -\frac{\pi}{2}$

(c) $\mathbf{r}_1 = (1-t)\mathbf{i} - t\mathbf{j}, 0 \le t \le 1,$ and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{dr_1}{dt} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{dr_1}{dt} = (2t-1) + (1-2t+2t^2) = 2t^2 \Rightarrow \ \text{Flow}_1 = \int_{c_1} \mathbf{F} \cdot \frac{dr_1}{dt} = \int_0^1 2t^2 dt = \frac{2}{3}; r_2 = -t\mathbf{i} + (t-1)\mathbf{j}, 0 \le t \le 1,\ \text{and} \mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{dr_2}{dt} = -\mathbf{i} + \mathbf{j}\ \text{and} \mathbf{F} = -\mathbf{i} - (t^2 + t^2 - 2t + 1)\mathbf{j} = -\mathbf{i} - (2t^2 - 2t + 1)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{dr_2}{dt} = 1 - (2t^2 - 2t + 1) = 2t - 2t^2 \Rightarrow \ \text{Flow}_2 = \int_{c_2} \mathbf{F} \cdot \frac{dr_2}{dt} = \int_0^1 (2t - 2t^2) dt = \left[t^2 - \frac{2}{3}t^3 \right]_0^1 = \frac{1}{3} \Rightarrow \text{Flow} = \text{Flow}_1 + \text{Flow}_2 = \frac{2}{3} + \frac{1}{3} = 1$

Exercises

Exercise 7.

1. *Find the circulation and flux of the field*

$$\mathbf{F} = -y^2\mathbf{i} + x^2\mathbf{j}$$

around and across the closed semicircular path that consists of the semicircular arc $\mathbf{r}_1(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ \ 0 \le t \le \pi$, *followed by the line segment* $\mathbf{r}_2(t) = t\mathbf{i},\ \ -a \le t \le a$.

2. *Find the circulation and flux of the fields* $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$ *and* $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$ *across the ellipse* $\mathbf{r}(t) = (\cos t)\mathbf{i} + (4\sin t)\mathbf{j}$, $0 \le t \le 2\pi$.

Solution for Exercise 7

1. $\mathbf{F}_1 = (-a^2\sin^2 t)\mathbf{i} + (a^2\cos^2 t)\mathbf{j}$, $\frac{dr_1}{dt} = (-a \ \sin \ t)\mathbf{i} + (a \ \cos \ t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{dr_1}{dt} = a^3\sin^3 t + a^3\cos^3 t \Rightarrow \text{Circ}_1 = \int_0^\pi (a^3\sin^3 t + a^3\cos^3 t)dt = \frac{4}{3}a^3$; $M_1 = -a^2\sin^2 t$, $N_1 = a^2\cos^2 t$, $dy = a \ \cos \ t \ dt$, $dx = -a \ \sin \ t \ dt \Rightarrow \text{Flux}_1 = \int_c M_1 dy - N_1 dx = \int_0^\pi (-a^3\cos \ t \ \sin^2 t + a^3\sin \ t \ \cos^2 t)dt = \frac{2}{3}a^3$; $\mathbf{F}_2 = t^2\mathbf{j}$, $\frac{dr_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{dr_2}{dt} = 0 \Rightarrow \text{Circ}_2 = 0$; $M_2 = 0$, $N_2 = t^2$, $dy = 0$, $dx = dt \Rightarrow \text{Flux}_2 = \int_c M_2 dy - N_2 dx = \int_{-a}^a -t^2 dt = -\frac{2}{3}a^3$; therefore, $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = \frac{4}{3}a^3$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$.
2. $\mathbf{r} = (\cos \ t)\mathbf{i} + (4 \ \sin \ t)\mathbf{j}, 0 \leq t \leq 2\pi \Rightarrow \frac{dr}{dt} = (-\sin \ t)\mathbf{i} + (4\cos \ t)\mathbf{j}$, $\mathbf{F}_1 = (\cos \ t)\mathbf{i} + (4\sin \ t)\mathbf{j}$, and $\mathbf{F}_2 = (-4\sin \ t)\mathbf{i} + (\cos \ t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{dr}{dt} = 15\sin \ t \ \cos \ t$ and $\mathbf{F}_2 \cdot \frac{dr}{dt} = 4 \Rightarrow \text{Circ}_1 = \int_0^{2\pi} 15 \ \sin \ t \ \cos \ t \ dt = \left[\frac{15}{2}\sin^2 t\right]_0^{2\pi} = 0$ and $\text{Circ}_2 = \int_0^{2\pi} 4 \ dt = 8\pi$; $\mathbf{n} = \left(\frac{4}{\sqrt{17}}\cos \ t\right)\mathbf{i} + \left(\frac{1}{\sqrt{17}}\sin \ t\right)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n} = \frac{4}{\sqrt{17}}\cos^2 t + \frac{4}{\sqrt{17}}\sin^2 t$ and
$\mathbf{F}_2 \cdot \mathbf{n} = \frac{-15}{\sqrt{17}}\sin t\cos t$. $\text{Flux}_1 = \int_0^{2\pi} \frac{4}{\sqrt{17}}\sqrt{17} \ dt = 8\pi$ and
$\text{Flux}_2 = \int_0^{2\pi} (\frac{-15}{\sqrt{17}}\sin t\cos t)\sqrt{17} \ dt = 0$.

Exercise 8.

Find the flux of the field $\mathsf{F} = (x+y)\mathsf{i} - (x^2+y^2)\mathsf{j}$ *outward across the triangle with vertices* $(1,0), (0,1), (-1,0)$.

Solution for Exercise 8

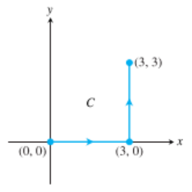
From $(1, 0)$ to $(0, 1)$:
$\mathbf{r}_1 = (1, -1)\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1,$ and $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j} \Rightarrow \frac{dr_1}{dt} = -\mathbf{i} + \mathbf{j}, \mathbf{F} = \mathbf{i} - (1 - 2t + 2t^2)\mathbf{j},$ and$\mathbf{n}_1|v_1| = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_1|v_1| = 2t - 2t^2 \Rightarrow \text{Flux}_1 = \int_0^1 (2t - 2t^2)dt = \left[t^2 - \frac{2}{3}t^3\right]_0^1 = \frac{1}{3};$

From $(0, 1)$ to $(-1, 0)$:
$\mathbf{r}_2 = -t\mathbf{i} + (1 - t)\mathbf{j}, 0 \leq t \leq 1,$ and$\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j} \Rightarrow \frac{dr_2}{dt} = -\mathbf{i} - \mathbf{j}, \mathbf{F} = (1 - 2t)\mathbf{i} - (1 - 2t + 2t^2)\mathbf{j},$ and $\mathbf{n}_2|v_2| = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_2|v_2| = (2t - 1) + (-1 + 2t - 2t^2) = -2 + 4t - 2t^2 \Rightarrow \text{Flux}_2 = \int_0^1 (-2 + 4t - 2t^2)dt = \left[-2t + 2t^2 - \frac{2}{3}t^3\right]_0^1 = -\frac{2}{3};$

From $(-1, 0)$ to $(1, 0)$:
$\mathbf{r}_3 = -(-1 + 2t)\mathbf{i} + 0 \leq t \leq 1,$ and $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j} \Rightarrow \frac{dr_3}{dt} = 2\mathbf{i}, \mathbf{F} = (-1 + 2t)\mathbf{i} - (1 - 4t + 4t^2)\mathbf{j},$ and $\mathbf{n}_3|v_3| = -2\mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_3|v_3| = 2(1 - 4t + 4t^2) \Rightarrow \text{Flux}_3 = 2\int_0^1 (1 - 4t + 4t^2)dt = 2\left[t - 2t^2 + \frac{4}{3}t^3\right]_0^1 = \frac{2}{3}; \Rightarrow \text{Flux} = \text{Flux}_1 + \text{Flux}_2 + \text{Flux}_3 = \frac{1}{3} - \frac{2}{3} + \frac{2}{3} = \frac{1}{3}$

Exercise 9.

1. $\int_C (x^2 + y^2)dy$, *where C is given in the accompanying figure,*



2. *Along the curve* $r(t) = (\cos t)\mathrm{i} + (\sin t)\mathrm{j} - (\cos t)\mathrm{k}, 0 \leq t \leq \pi$, *evaluate each of the following integrals.*

   (a) $\displaystyle\int_c xz\ dx$ (b) $\displaystyle\int_c xz\ dy$ (c) $\displaystyle\int_C xyz\ dz$

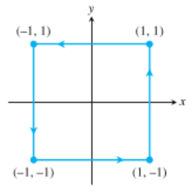
3. *Evaluate* $\int_C \mathrm{F.T}\ ds$ *for the vector field* $\mathrm{F} = x^2\mathrm{i} - y\mathrm{j}$ *along the curve* $x = y^2$ *from* $(4, 2)$ *to* $(1, -1)$.

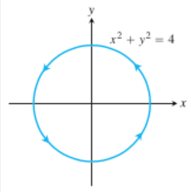
Solution for Exercise 9

1. $C_1 : x = t, y = 0, 0 \le t \le 3 \Rightarrow dy = 0$; $C_2 : x = 3, y = t, 0 \le t \le 3 \Rightarrow dy = dt \Rightarrow$ $\int_c (x^2 + y^2)dy = \int_{c_1}(x^2 + y^2)dx + \int_{c_2}(x^2 + y^2)dx = \int_0^3 (t^2 + 0^2) \cdot 0 + \int_0^3 (3^2 + t^2)dt =$ $\int_0^3 (9 + t^2)dt = [9t + \frac{1}{3}t^3]_0^3 = 36$
2. $\mathbf{r}(t) = (\cos\ t)\mathbf{i} + (\sin\ t)\mathbf{j} - (\cos\ t)\mathbf{k}, 0 \le t \le \pi \Rightarrow dx = -\sin\ t\ dt, dy = \cos\ t\ dt, dz = \sin\ t\ dt$
   (a) $\int_c x\ z\ dx = \int_0^\pi (\cos\ t)(-\cos\ t)(-\sin\ t)dt = \int_0^\pi \cos^2 t \sin\ t dt = \left[-\frac{1}{3}(\cos\ t)^3\right]_0^\pi = \frac{2}{3}$
   (b) $\int_c x\ z\ dy = \int_0^\pi (\cos\ t)(-\cos\ t)(\cos\ t)dt = -\int_0^\pi \cos^3 t\ dt = -\int_0^\pi (1 - \sin^2 t)\cos\ t\ dt = \left[\frac{1}{3}(\sin\ t)^3 - \sin\ t\right]_0^\pi = 0$
   (c) $\int_c x\ y\ z\ dz = \int_0^\pi (\cos\ t)(\sin\ t)(-\cos\ t)(\sin\ t)dt = -\int_0^\pi \cos^2 t\ \sin^2 t\ dt = -\frac{1}{4}\int_0^\pi \sin^2 2t\ dt = -\frac{1}{4}\int_0^\pi \frac{1-\cos\ 4t}{2}dt = -\frac{1}{8}\int_0^\pi (1 - \cos\ 4t)dt = \left[-\frac{1}{8}t + \frac{1}{32}\sin 4t\right]_0^\pi = -\frac{\pi}{8}$
3. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = y^2\mathbf{i} + y\mathbf{j}, 2 \ge y \ge -1,$ and $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} = y^4\mathbf{i} - y\mathbf{j} \Rightarrow \frac{dr}{dy} = 2y\mathbf{i} + \mathbf{j}$ and $\mathbf{F} \cdot \frac{dr}{dy} = 2y^5 - y \Rightarrow \int_c \mathbf{F} \cdot \mathbf{T}\ ds \int_2^{-1} \mathbf{F} \cdot \frac{dr}{dy}dy = \int_2^{-1}(2y^5 - y)dy = \left[\frac{1}{3}y^6 - \frac{1}{2}y^2\right]_2^{-1} = \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{64}{3} - \frac{4}{2}\right) = \frac{3}{2} - \frac{63}{3} = -\frac{39}{2}$

Exercise 10.

1. *Find the circulation of the field* $\mathbf{F} = y\mathbf{i} + (x + 2y)\mathbf{j}$ *around each of the following closed paths.*

(a) 

(b) 

(c) *Use any closed path different from parts (a) and (b).*

Solution for Exercise 10

(a) $C_1 : \mathbf{r}(t) = (1-t)\mathbf{i}+\mathbf{j}, 0 \le t \le 2 \Rightarrow \frac{dr}{dt} = -\mathbf{i} \Rightarrow \mathbf{F}\cdot\frac{dr}{dt} = ((1)\mathbf{i}+((1-t)+2(1)\mathbf{j})\cdot(-\mathbf{i}) = -1;$
$C_2 : \mathbf{r}(t) = -\mathbf{i} + (1-t)\mathbf{j}, 0 \le t \le 2 \Rightarrow \frac{dr}{dt} = -\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{dr}{dt} =$
$((1-t)\mathbf{i} + ((-1)+2(1-t))\mathbf{j})\cdot(-\mathbf{j}) = 2t-1;$
$C_3 : \mathbf{r}(t) = (t-1)\mathbf{i}\text{-}\mathbf{j}, 0 \le t \le 2 \Rightarrow \frac{dr}{dt} = \mathbf{i} \Rightarrow \mathbf{F}\cdot\frac{dr}{dt} = ((-1)\mathbf{i}+((t-1)+2(-1))\mathbf{j})\cdot(\mathbf{i}) = -1;$
$C_4 : \mathbf{r}(t) = \mathbf{i}+(t-1)\mathbf{j}, 0 \le t \le 2 \Rightarrow \frac{dr}{dt} = \mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{dr}{dt} = ((t-1)\mathbf{i}+((1)+2(t-1))\mathbf{j})\cdot(\mathbf{j}) = 2t-1;$
$\Rightarrow \text{Flow} = \int_c \mathbf{F}\cdot\frac{dr}{dt}dt = \int_{c_1}\mathbf{F}\cdot\frac{dr}{dt}dt + \int_{c_2}\mathbf{F}\cdot\frac{dr}{dt}dt + \int_{c_3}\mathbf{F}\cdot\frac{dr}{dt}dt + \int_{c_4}\mathbf{F}\cdot\frac{dr}{dt}dt$
$= \int_0^2(-1)dt + \int_0^2(2t-1)dt + \int_0^2(-1)dt + \int_0^2(2t-1)dt =$
$[-t]_0^2 + [t^2-t]_0^2 + [-t]_0^2 + [t^2-t]_0^2 = -2+2-2+2 = 0$

(b) $x^2+y^2 = 4 \Rightarrow \mathbf{r}(t) = (2\cos\ t)\mathbf{i} + (2\sin t)\mathbf{j}, 0 \le t \le 2\pi \Rightarrow \frac{dr}{dt} = (-2\sin\ t)\mathbf{i} + (2\cos\ t)\mathbf{j} \Rightarrow$
$\mathbf{F}\cdot\frac{dr}{dt} = ((2\sin\ t)\mathbf{i} + (2\cos\ t + 2(2\sin\ t))\mathbf{j})\cdot((-2\sin\ t)\mathbf{i} + (2\cos\ t)\mathbf{j}) =$
$-4\sin^2 t + 4cos^2 t + 8\sin\ t\ \cos\ t = 4\cos\ 2t + 4\sin\ 2t \Rightarrow \text{Flow} = \int_c \text{F}\cdot\frac{dr}{dt} =$
$\int_0^{2\pi}(4\cos\ 2t + 4\sin\ 2t)dt = [2\sin\ 2t - 2\cos\ 2t]_0^{2\pi} = 0$

(c) Answer will vary, one possible path is:
$C_1 : \mathbf{r}(t) = t\mathbf{i}, 0 \le t \le 1 \Rightarrow \frac{dr}{dt} = \mathbf{i} \Rightarrow \mathbf{F}\cdot\frac{dr}{dt} = ((0)\mathbf{i} + (t+2(1)\mathbf{j})\cdot(\mathbf{i}) = 0;$
$C_2 : \mathbf{r}(t) = (1-t)\mathbf{i}+t\mathbf{j}, 0 \le t \le 1 \Rightarrow \frac{dr}{dt} = -\mathbf{i}+\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{dr}{dt} = (t\mathbf{i}+((1-t)+2t)\mathbf{j})\cdot(-\mathbf{i}+\mathbf{j}) = 1;$
$C_3 : \mathbf{r}(t) = (1-t)\mathbf{j}, 0 \le t \le 1 \Rightarrow \frac{dr}{dt} = -\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{dr}{dt} = ((1-t)\mathbf{i}+(0+2(1-t))\mathbf{j})\cdot(-\mathbf{j}) = 2t-1;$

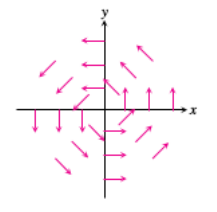
Spin field

Exercise 11.

Draw the spin field

$$F = -\frac{y}{\sqrt{x^2+y^2}}\mathrm{i} + \frac{x}{\sqrt{x^2+y^2}}\mathrm{j}$$

(see the figure) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 4$.



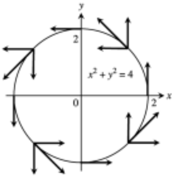
Solution for Exercise 11

$\mathbf{F} = -\frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$ on $x^2 + y^2 = 4$;

at $(2, 0), \mathbf{F} = \mathbf{j}$; at $(0, 2), \mathbf{F} = -\mathbf{i}$; at$(-2, 0), \mathbf{F} = -\mathbf{j}$; at $(0, -2), \mathbf{F} = \mathbf{i}$;

at $(\sqrt{2}, \sqrt{2}), \mathbf{F} = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$; at $(\sqrt{2}, -\sqrt{2}), \mathbf{F} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$;

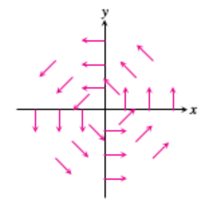
at$(-\sqrt{2}, \sqrt{2}), \mathbf{F} = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$; at $(-\sqrt{2}, -\sqrt{2}), \mathbf{F} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$



Exercise 12.

(a) *Find a field* $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ *in the* xy*-plane with the property that at any point* $(a, b) \neq (0, 0)$, $\mathbf{G}$ *is a vector of magnitude* $\sqrt{(a^2 + b^2)}$ *tangent to the circle* $x^2 + y^2 = a^2 + b^2$ *and pointing in the counterclockwise direction. (The field is undefined at* $(0, 0)$*.)*

(b) *How is* $\mathbf{G}$ *related to the spin field* $\mathbf{F}$ *in the following figure?*



Solution for Exercise 12

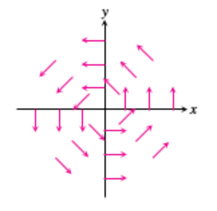
(a) $\mathbf{G} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is to have a magnitude $\sqrt{a^2+b^2}$ and to be tangent to $x^2+y^2=a^2+b^2$ in a counterclockwise direction. Thus $x^2+y^2=a^2+b^2 \Rightarrow 2x+2yy'=0 \Rightarrow y'=-\frac{x}{y}$ is the slope of the tangent line at any point on the circle $\Rightarrow y'=-\frac{a}{b}$ at(a,b). Let $v=-b\mathbf{i}+a\mathbf{j} \Rightarrow |\mathbf{v}|=\sqrt{a^2+b^2}$, with $\mathbf{v}$ in a counterclockwise direction and tangent to the circle. Then let $P(x,y)=-y$ and $Q(x,y)=x \Rightarrow \mathbf{G}=-y\mathbf{i}+x\mathbf{j} \Rightarrow$ for (a,b)on $x^2+y^2=a^2+b^2$ we have $\mathbf{G}=-b\mathbf{i}+a\mathbf{j}$and $|\mathbf{G}|=\sqrt{a^2+b^2}$.

(b) $\mathbf{G}=(\sqrt{x^2}+y^2)\mathbf{F}=(\sqrt{a^2}b^2)\mathbf{F}$.

Exercise 13.

(a) *Find a field* $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ *in the xy-plane with the property that at any point* $(a, b) \neq (0, 0)$, $\mathbf{G}$ *is a unit vector tangent to the circle* $x^2 + y^2 = a^2 + b^2$ *and pointing in the clockwise direction.*

(b) *How is* $\mathbf{G}$ *related to the spin field* $\mathbf{F}$ *in the following figure?*



Solution for Exercise 13

(a) From Exercise 12, part a, $-y\mathbf{i} + x\mathbf{j}$ is a vector tangent to the circle and pointing in a counterclockwise direction $\Rightarrow y\mathbf{i} - x\mathbf{j}$ is a vector tangent to the circle pointing in a clockwise direction $\Rightarrow \mathbf{G} = \frac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2+y^2}}$ is a unit vector tangent to the circle and pointing in a clockwise direction.

(b) $\mathbf{G} = -\mathbf{F}$

Exercise 14.

1. Unit vectors pointing toward the origin : *Find a field* $F = M(x, y)i + N(x, y)j$ *in the xy-plane with the property that at each point* $(x, y) \neq (0, 0)$, F *is a unit vector pointing toward the origin. (The field is undefined at* $(0, 0)$*.)*
2. Two central fields : *Find a field* $F = M(x, y)i + N(x, y)j$ *in the xy-plane with the property that at each point* $(x, y) \neq (0, 0)$, F *points toward the origin and* $|F|$ *is*
   (a) *the distance from* (x, y) *to the origin,*
   (b) *inversely proportional to the distance from* (x, y) *to the origin. (The field is undefined at* $(0, 0)$*.)*

Solution for Exercise 14

1. The slope of the line through (x, y) and the origin is $\frac{y}{x} \Rightarrow \mathbf{v} = x\mathbf{i} + y\mathbf{j}$ is a vector parallel to that line and pointing away from the origin $\Rightarrow \mathbf{F} = -\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ is the unit vector pointing toward the origin.
2. (a) From the above exercise, $-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ is a unit vector through (x, y) pointing toward the origin and we want $|\mathbf{F}|$ to have magnitude $\sqrt{x^2+y^2} \Rightarrow \mathbf{F} = \sqrt{x^2+y^2}\left(-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}\right) = -x\mathbf{i} - y\mathbf{j}$

   (b) We want $|\mathbf{F}| = \frac{c}{\sqrt{x^2+y^2}}$ where $C \neq 0$ is a constant $\Rightarrow \mathbf{F} = \frac{C}{\sqrt{x^2+y^2}}\left(-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}\right) = -C\left(\frac{x\mathbf{i}+y\mathbf{j}}{x^2+y^2}\right)$.

Exercise 15.

Work and area : *Suspose that* $f(t)$ *is differentiable and positive for* $a \leq t \leq b$ *Let* C *be the path* $\mathbf{r}(t) = t\mathbf{i} + f(t)\mathbf{j}$, $a \leq t \leq b$, *and* $\mathbf{F} = y\mathbf{i}$. *Is there any relation between the value of the work integral*

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

and the area of the region bounded by the t*-axis, the graph of* f*, and the lines* $t = a$ *and* $t = b$*? Give reasons for your answer.*

Solution for Exercise 16

Yes. The work and area have the same numerical value because work $= \int_C \mathbf{F} \cdot dr = \int_c y\mathbf{i} \cdot d\mathbf{r}$

$$= \int_b^a [f(t)\mathbf{i}] \cdot \left[\mathbf{i} + \frac{df}{dt}\mathbf{j}\right] dt$$ [On the path, y equals f(t)]

$$= \int_a^b f(t)dt = \text{Area under the curve}$$ [because$f(t) > 0$]

Exercise 16.

F *is the velocity field of a fluid flowing through a region in space. Find the flow along the given curve in the direction of increasing* t.

(a) $\mathrm{F} = x^2\mathrm{i} + yz\mathrm{j} + y^2\mathrm{k}$
$\mathrm{r}(t) = 3t\mathrm{j} + 4t\mathrm{k}, \quad 0 \leq t \leq 1$

(b) $\mathrm{F} = -y\mathrm{i} + x\mathrm{j} + 2\mathrm{k}$
$\mathrm{r}(t) = (-2\cos t)\mathrm{i} + (2\sin t)\mathrm{j} + 2t\mathrm{k}, \quad 0 \leq t \leq 2\pi$

Solution for Exercise 16

(a) $\mathbf{F} = 12t^2\mathbf{j} + 9t^2\mathbf{k}$
Flow $= \int_0^1 72t^2\ dt = 24.$

(b) $\mathbf{F} = (-2\sin t)\mathbf{i} - (2\cos t)x\mathbf{j} + 2\mathbf{k}$
Flow $= 0.$

Exercise 17.

Work done by a radial force with constant magnitude: *A particle moves along the smooth curve* $y = f(x)$ *from* $(a, f(a))$ *to* $(b, f(b))$*. The force moving the particle has constant magnitude* k *and always points away from the origin. Show that the work done by the force is*

$$\int_C \mathrm{F} \cdot \mathrm{T}\, ds = k \left[(b^2 + (f(b))^2)^{1/2} - (a^2 + (f(a))^2)^{1/2} \right].$$

Solution for Exercise 17

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + f(x)\mathbf{j} \Rightarrow \frac{dr}{dx} = \mathbf{i} + f'(x)\mathbf{j}$; $\mathbf{F} = \frac{k}{\sqrt{x^2+y^2}}(x\mathbf{i} + y\mathbf{j})$ has constant magnitude k and points away from the origin

$\Rightarrow \mathbf{F} \cdot \frac{dr}{dx} = \frac{kx}{\sqrt{x^2+y^2}} + \frac{k \cdot y \cdot f'(x)}{\sqrt{x^2+y^2}} = \frac{kx + k \cdot f(x) \cdot f'(x)}{\sqrt{x^2+[f(x)]^2}} = k \ \frac{d}{dx}\sqrt{x^2 + [f(x)]^2}$, by the chain rule

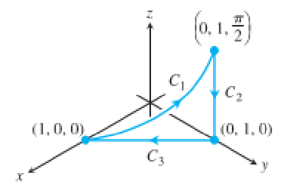
$\Rightarrow \int_c \mathbf{F} \cdot \mathbf{T} \ ds = \int_c \mathbf{F} \cdot \frac{dr}{dx} dx = \int_a^b k \frac{d}{dx}\sqrt{x^2 + [f(x)]^2} dx = k\left[\sqrt{x^2 + +[f(x)]^2}\right]_a^b$

$= k(\sqrt{b^2 + [f(b)]^2} - \sqrt{a^2 + [f(a)]^2})$, as claimed.

Exercise 18.

Circulation: *Find the circulation of* $\mathbf{F} = 2x\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}$ *around the closed path consisting of the following three curves traversed in the direction of increasing* t.

$$C_1 : \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi/2$$
$$C_2 : \mathbf{r}(t) = \mathbf{j} + (\pi/2)(1 - t)\mathbf{k}, \quad 0 \le t \le 1$$
$$C_3 : \mathbf{r}(t) = t\mathbf{i} + (1 - t)\mathbf{j}, \quad 0 \le t \le 1$$



Solution for Exercise 18

$C_1 : \mathbf{r} = (\cos\ t)\mathbf{i} + (\sin\ t)\mathbf{j} + t\mathbf{k}, 0 \le t \le \frac{\pi}{2} \Rightarrow \mathbf{F} = (2\ \cos\ t)\mathbf{i} + 2t\mathbf{j} + (2\ \sin\ t)\mathbf{k}$ and $\frac{dr}{dt} = (-\sin\ t)\mathbf{i} + (\cos\ t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = -2\ \cos\ t\ \sin\ t + 2t\ \cos\ t + 2\ \sin\ t = -\sin\ 2t + 2t\ \cos\ t + 2\ \sin t$

$\Rightarrow \text{Flow}_1 = \int_0^{\pi/2} (-\sin\ 2t + 2t\ \cos\ t + 2\ \sin\ t) dt = \left[\frac{1}{2} \cos\ 2t + 2t\ \sin\ t + 2\ \cos\ t - 2\ \cos\ t\right]_0^{\pi/2} = -1 + \pi;$
$C_2 : \mathbf{r} = \mathbf{j} + \frac{\pi}{2}(1-t)\mathbf{k}, 0 \le t \le 1 \Rightarrow \mathbf{F} = \pi(1-t)\mathbf{j} + 2\mathbf{k}$ and $\frac{dr}{dt} = -\frac{\pi}{2}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = -\pi$

$\Rightarrow \text{Flow}_2 = \int_0^1 -\pi\ dt = [-\pi t]_0^1 = -\pi;$

$C_3 : \mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j}, 0 \le t \le 1 \Rightarrow \mathbf{F} = 2t\mathbf{i} + 2(1-t)\mathbf{k}$ and $\frac{dr}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = 2t$

$\Rightarrow \text{Flow}_3 = \int_0^1 2t\ dt = [t^2]_0^1 = 1 \Rightarrow \text{Circulation} = (-1 + \pi) - \pi + 1 = 0$

Exercise 19.

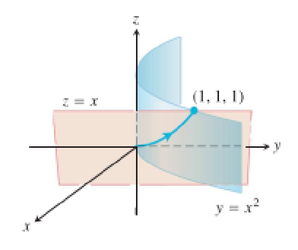
1. Zero circulation: *Let C be the ellipse in which the plane* $2x + 3y - z = 0$ *meets the cylinder* $x^2 + y^2 = 12$. *Show, with-out evaluating either line integral directly, that the circulation of the field* $\mathrm{F} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$ *around C in either direction is zero.*
2. Flow of a gradient field: *Find the flow of the field* $\mathrm{F} = \nabla(xy^2z^3)$*:*
   (a) *Once around the curve C in Exercise 1, clockwise as viewed from above*
   (b) *Along the line segment from* $(1, 1, 1)$ *to* $(2, 1, -1)$.

Solution for Exercise 19

1. $\mathbf{F} \cdot \frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$, where
$f(x, y, z) = \frac{1}{2}(x^2 + y^2 + x^2) \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = \frac{d}{dt}(f(\mathbf{r}(t)))$ by the chain rule
$\Rightarrow$ Circulation $= \int_c \mathbf{F}.\frac{dr}{dt}dt = \int_a^b \frac{d}{dt}(f(\mathbf{r}(t)))dt = f(\mathbf{r}(b)) - f(r(a))$. Since C is an entire ellipse, $\mathbf{r}(b) = \mathbf{r}(a)$,thus the Circulation $= 0$.
2. (a) $\mathbf{F} = \nabla(xy^2z^3) \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial z}\frac{dz}{dt} = \frac{df}{dt}$, where
$f(x, y, z) = xy^2z^3 \Rightarrow \oint_c \mathbf{F} \cdot \frac{dr}{dt}dt = \int_a^b \frac{d}{dt}(f(\mathbf{r}(t)))dt = dt =$
$f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0$ since C is an entire ellipse.
   (b) $\int_c \mathbf{F} \cdot \frac{dr}{dt} = \int_{(1,1,1)}^{(2,1,-1)} \frac{d}{dt}(xy^2z^3)dt = [xy^2z^3]_{(1,1,1)}^{(2,1,-1)} =$
$(2)(1)^2(-1)^3 - (1)(1)^2(1)^3 = -2 - 1 = -3$

Exercise 20.

Flow along a curve: *The field* $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$ *is the velocity field of a flow in space. Find the flow from* $(0,0,0)$ *to* $(1,1,1)$ *along the curve of intersection of the cylinder* $y = x^2$ *and the plane* $z = x$*. (Hint: Use* $t = x$ *as the parameter.)*



Solution for Exercise 20

Let $r = t$ be the parameter $\Rightarrow y = x^2 = t^2$ and $z = x = t \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, 0 \leq t \ \leq 1$ from $(0,0,0)$ to $(1,1,1) \Rightarrow \frac{dr}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$ and $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = t^3 + 2t^3 - t^3 = 2t^3 \Rightarrow$ Flow $= \int_0^1 2t^3 dt = \frac{1}{2}$

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